POLYNOMIAL FUNCTIONS

Precalculus
Chapter 2

- This Slideshow was developed to accompany the textbook
  - *Precalculus*
  - *By Richard Wright*
  - *https://www.andrews.edu/~rwright/Precalculus-RLW/Text/TOC.html*
- Some examples and diagrams are taken from the textbook.

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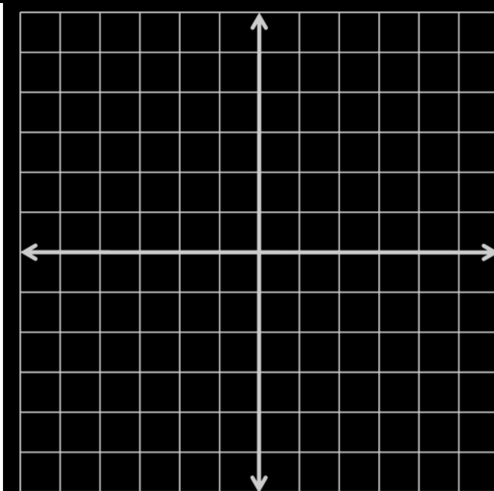
2-01 COMPLEX NUMBERS

In this section, you will:

- Express square roots of negative numbers as multiples of *i*.
- Plot complex numbers on the complex plane.
- Add and subtract complex numbers.
- Multiply and divide complex numbers.

- Imaginary Unit
  - $i = \sqrt{-1}$
  - $i^2 = -1$
- Complex Number
  - $a + bi$
  - a is real part
  - bi is imaginary part

- Graphing complex points
- Complex plane
  - Horizontal axis = Real
  - Vertical axis=Imaginary
- Graph
  - $3 + 2i$
  - $-2 - 4i$



- Operations
  - Add/Subtract
  - Combine like terms
- $(2 + 3i) + (6 - 2i)$

- Multiplication
  - i^2 becomes -1
- $(2 + 3i)(6 - 7i)$

$8 - 4i$

$12 - 14i + 18i - 21i^2$

$12 - 14i + 18i + 21$

$33 + 4i$

- $(3+i)(3-i)$

- Complex Conjugates
- $a+bi$ and $a-bi$
- When complex conjugates are multiplied, the product is real

10

- Division
  - Multiply by conjugate of denominator
- $\frac{2-3i}{4+6i}$

$$\frac{(2-3i)}{(4+6i)}\frac{(4-6i)}{(4-6i)}$$

$$\frac{8-12i-12i+18i^2}{16-24i+24i-36i^2}$$

$$\frac{8-12i-12i+18(-1)}{16-24i+24i-36(-1)}$$

$$\frac{-10-24i}{52}$$

$$-\frac{5}{26}-\frac{6}{13}i$$

- $(5 - i)^2$

$$(5 - i)(5 - i)$$
$$25 - 5i - 5i + i^2$$
$$24 - 10i$$

- $\sqrt{-14}\sqrt{-2}$
- $\sqrt{-27} - \sqrt{-12}$

$$i\sqrt{14} \bullet i\sqrt{2}$$
$$i^2\sqrt{28}$$
$$-1\sqrt{4}\sqrt{7}$$
$$-2\sqrt{7}$$

$$i\sqrt{9}\sqrt{3} - i\sqrt{4}\sqrt{3}$$
$$3i\sqrt{3} - 2i\sqrt{3}$$
$$i\sqrt{3}$$

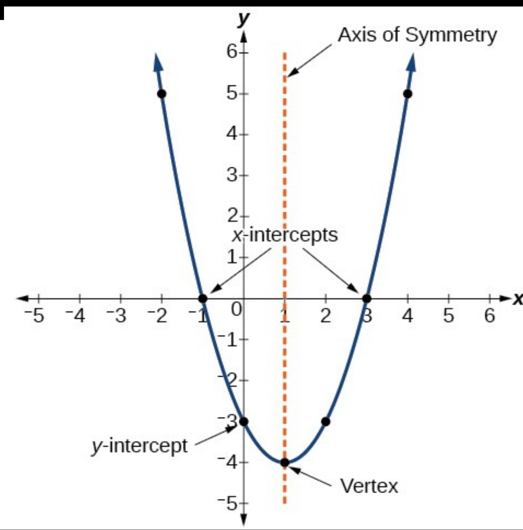
2-02 QUADRATIC EQUATIONS

In this section, you will:

- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.
- Determine a quadratic function’s minimum or maximum value.
- Solve problems involving a quadratic function’s minimum or maximum value.

- $f(x) = ax^2 + bx + c$

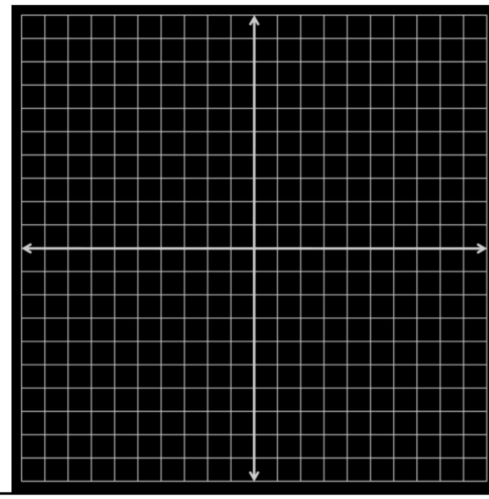
- $|a| > 1$ → thinner
- $0 < |a| < 1$ → wider
- $a < 0$ → reflected over x-axis "opens down"
- $a > 0$ → "opens up"



- Standard Form
- $f(x) = a(x-h)^2 + k$
  - Vertex (h, k)
  - Axis of symmetry $x = h$
- To graph
  1. Find vertex
  2. Make table around vertex
  3. Draw parabola

- General Form
- $f(x) = ax^2 + bx + c$
  - Vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$
  - Axis $x = -\frac{b}{2a}$

- Graph $f(x) = x^2 - 10x + 25$ and identify the vertex and axis of symmetry



Vertex: (5, 0)
Axis of symmetry: $x = 5$

- Write the standard form of the equation of parabola with vertex $(-4, 11)$ and passes through $(-6, 15)$

$$y = a(x + 4)^2 + 11$$
$$15 = a(-6 + 4)^2 + 11$$
$$4 = a4$$
$$a = 1$$
$$y = (x + 4)^2 + 11$$

- Maximum and minimum
- Occurs at the vertex

- Quadratic formula
- $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- Solve $8x^2 + 14x + 9 = 0$

$$x = \frac{-14 \pm \sqrt{14^2 - 4(8)(9)}}{2(8)}$$

$$x = \frac{-14 \pm \sqrt{-92}}{16}$$

$$x = \frac{-14 \pm 2\sqrt{23}i}{16}$$

$$x = -\frac{7}{8} \pm \frac{\sqrt{23}}{8}i$$

2-03 POLYNOMIAL EQUATIONS

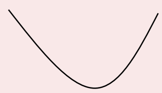
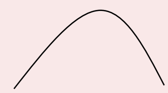
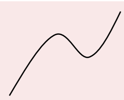
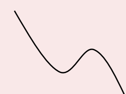
In this section, you will:

- Identify polynomial functions.
- Identify the end behavior.
- Graph polynomial functions.
- Write polynomial functions.

- Polynomial Function
- $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$

- a_n are coefficients
- $a_n x^n$ are terms
- a_0 is constant term
- Degree is highest exponent
- Leading coefficient is coefficient of term with highest exponent
- Graphs are continuous, smooth, rounded turns

- End Behavior
  - Polynomial functions always go towards ∞ or -∞ at either end of the graph

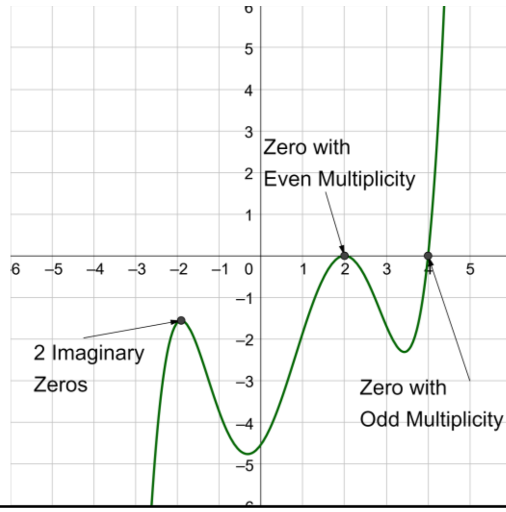
	Leading Coefficient +	**Leading Coefficient -**
Even Degree		
Odd Degree		

- What is the end behavior of $f(x) = \frac{1}{3}x^3 + 5x$?

Odd degree and positive leading coefficient
Falls to left and rises to right

- Zeros
- If $x = a$ is a zero of $f(x)$
  - $x = a$ is a solution to $f(x) = 0$
  - $(a, 0)$ is x-intercept
  - $(x - a)$ is a factor of $f(x)$

- Let $f(x)$ have degree n
  - At most n zeros (can be repeated)
  - At most $n - 1$ turning points


Zero with
Even Multiplicity
2 Imaginary
Zeros
Zero with
Odd Multiplicity

For $g(t) = t^5 - 6t^3 + 9t$

a. Find all zeros
b. Find multiplicity of zeros
c. Graph

a. Factor

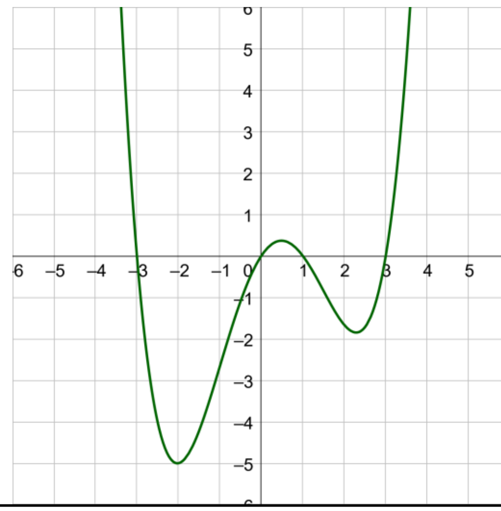
$0, \sqrt{3}, -\sqrt{3}$

b. 0: 1; $\sqrt{3}$: 2; $-\sqrt{3}$: 2
c. Very tall graph

- Find the intercepts of $f(x) = x(x + 2)(x - 3)$

X-int: 0, -2, 3 (let y=0 and solve for x)
Y-int: 0 (let x=0 and solve for y)

- Determine the least possible degree of the polynomial function shown.



4 zero so n = 4
3 turning points so n-1=3, so n=4

2-04 DIVIDING POLYNOMIALS

In this section, you will:

- Divide polynomials with long division.
- Divide polynomials with synthetic division.

- Long Division
  - Done like long division with numbers
- $\frac{y^4+2y^2-y+5}{y^2-y+1}$

Divide the 1st terms

Multiply

Subtract

- $$\begin{array}{r} \boxed{y^2} \boxed{+y} \boxed{+2} + \frac{3}{y^2-y+1} \\ \boxed{y^2-y+1}\overline{)y^4+0y^3+2y^2-y+5} \\ \underline{y^4-y^3+y^2} \\ y^3+y^2-y \\ \underline{y^3-y^2+y} \\ 2y^2-2y+5 \\ \underline{2y^2-2y+2} \\ 3 \end{array}$$

- $\frac{x^3+4x^2-3x+10}{x+2}$

Divide the 1st terms

Multiply

Subtract

- $$
\begin{array}{r|l}
 & x^2 + 2x - 7 + \frac{24}{x+2} \\
x+2) & x^3 + 4x^2 - 3x + 10 \\
 & \underline{x^3 + 2x^2} \\
 & 2x^2 - 3x \\
 & \underline{2x^2 + 4x} \\
 & -7x + 10 \\
 & \underline{-7x - 14} \\
 & 24
\end{array}
$$

- Synthetic Division
  - Shortened form of long division for dividing by a **binomial**
  - Only when dividing by $(x - r)$

- Synthetic Division
  - $(-5x^5 - 21x^4 - 3x^3 + 4x^2 + 2x + 2)/(x + 4)$

Coefficients with placeholders

-4	-5	-21	-3	4	2	2
		20	4	-4	0	-8
	-5	-1	1	0	2	$\boxed{-6}$

- The bottom row give the coefficients of the answer. This is called the depressed polynomial.
- The number in the box is the remainder.
- Start with an exponent one less than the original expression since you divided by *x*.

$$-5x^4 - x^3 + x^2 + 2 + \frac{-6}{x+4}$$

- $(y^5 + 32)(y + 2)^{-1}$

Coefficients with placeholders

-2	1	0	0	0	0	32
×	+	-2 +	4 +	-8 +	16 +	-32 +
	1	-2	4	-8	16	0

- The degree was 5, now the depressed polynomial is one less degree. It's degree is 4.

$$y^4 - 2y^3 + 4y^2 - 8y + 16$$

- Factor Theorem
  - If $f(x)$ is divided by $(x - k)$ and remainder is 0, then $(x - k)$ is a factor of $f(x)$

- Show that $(x+3)$ is a factor of $x^3 - 19x - 30$. Then find the remaining factors.

$$
\begin{array}{r|rrrr}
\underline{-3} & 1 & 0 & -19 & -30 \\
 & & -3 & 9 & \underline{30} \\
\hline
 & 1 & -3 & -10 & |\underline{0}
\end{array}
$$

$$x^2 - 3x - 10$$
$$(x-5)(x+2)$$

Ans: $(x+3)(x-5)(x+2)$

2-05 RATIONAL ZEROS OF POLYNOMIAL FUNCTIONS

In this section, you will:

- Evaluate a polynomial using the Remainder Theorem.
- Use the Factor Theorem to solve a polynomial equation.
- Use the Rational Zero Theorem to find rational zeros.

- Remainder Theorem
  - If $f(x)$ is divided by $(x - k)$, then the remainder is $r = f(k)$

- Use the remainder theorem to evaluate $f(x) = 4x^3 + 10x^2 - 3x - 8$ for $f(-1)$

$$\begin{array}{r|rrrr} -1 & 4 & 10 & -3 & -8 \\ & & -4 & -6 & 9 \\ \hline & 4 & 6 & -9 & \underline{1} \end{array}$$

f(-1)=1

- Rational Zero Theorem
  - If polynomial

    $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$

    The rational zeros are in the form $\frac{p}{q}$

    where p = factors of a_0

    q = factors of a_n

- Find the rational zeros of $f(x) = x^3 - 5x^2 + 2x + 8$ given that $x + 1$ is a factor.

$\boldsymbol{p = \pm 1, \pm 2, \pm 4, \pm 8}$
$\boldsymbol{q = \pm 1}$
$\boldsymbol{\frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 8}$

$$\begin{array}{r|rrrr} -1 & 1 & -5 & 2 & 8 \\ & & -1 & 6 & -8 \\ \hline & 1 & -6 & 8 & \underline{0} \end{array}$$

Depressed Polynomial: $\boldsymbol{y = x^2 - 6x + 8}$
Since Quadratic, Solve $\boldsymbol{0 = x^2 - 6x + 8}$

$$\boldsymbol{0 = (x-2)(x-4)}$$
$$\boldsymbol{x - 2 = 0 \text{ or } x - 4 = 0}$$
$$\boldsymbol{x = 2 \text{ or } x = 4}$$

Zeros: −1, 2, 4

- Find the real zeros of $f(x) = x^3 - 7x^2 - 11x + 14$ given that $x + 2$ is a factor.

$\boldsymbol{p = \pm 1, \pm 2, \pm 7, \pm 14}$
$\boldsymbol{q = \pm 1}$
$\boldsymbol{\frac{p}{q} = \pm 1, \pm 2, \pm 7, \pm 14}$

−2 |	**1**	**−7**	**−11**	**14**
		−2	**18**	**−14**
	1	**−9**	**7**	**| 0**

Depressed Polynomial: $\boldsymbol{y = x^2 - 9x + 7}$
Since Quadratic, Solve $\boldsymbol{0 = x^2 - 9x + 7}$

$$\boldsymbol{x = \frac{9 \pm \sqrt{9^2 - 4(1)(7)}}{2(1)}}$$

$$\boldsymbol{x = \frac{9 \pm \sqrt{53}}{2}}$$

Zeros: −2, $\boldsymbol{\frac{9 \pm \sqrt{53}}{2}}$

2-06 ZEROS OF POLYNOMIAL FUNCTIONS

In this section, you will:

- Find zeros of a polynomial function.
- Use the Linear Factorization Theorem to find polynomials with given zeros.
- Use Descartes' Rule of Signs.

- Fundamental Theorem of Algebra
  - If $f(x)$ is polynomial of degree n, then there is at least 1 zero
  - There are exactly n zeros
  - There are n linear factors (Linear Factorization Theorem)

• Find all zeros of $f(x) = x^4 - 16$

Factor

$$0 = x^4 - 16$$
$$0 = (x^2 - 4)(x^2 + 4)$$
$$0 = (x - 2)(x + 2)(x^2 + 4)$$
$$x = 2, -2, 2i, -2i$$

- Find all the zeros of

$$f(x) = 2x^4 - 9x^3 - 18x^2 + 71x - 30$$

$$\boldsymbol{p = \pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30}$$
$$\boldsymbol{q = \pm 1, \pm 2}$$
$$\boldsymbol{\frac{p}{q} = \pm 1, \pm \frac{1}{2}, \pm 2, \pm 3, \pm \frac{3}{2}, \pm 5, \pm \frac{5}{2}, \pm 6, \pm 10, \pm 15, \pm \frac{15}{2}, \pm 30}$$

Pick one and divide

2 |	2	−9	−18	71	−30
		4	−10	−56	30
	2	−5	−28	15	| 0

Depressed polynomial: $\boldsymbol{2x^3 - 5x^2 - 28x + 15}$

Pick another possible zero and divide again

5 |	2	−5	−28	15
		10	25	−15
	2	5	−3	| 0

Depressed polynomial is quadratic

$$\boldsymbol{2x^2 + 5x - 3 = 0}$$

$$(2x-1)(x+3)=0$$
$$2x-1=0 \ or \ x+3=0$$
$$x=\frac{1}{2},-3$$

Zeros: $-3, \frac{1}{2}, 2, 5$

- Descartes's Rule of Signs
- Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$
  - The number of positive real zeros is equal to the number of variations in sign of $f(x)$ or less by even integer
  - The number of negative real zeros is equal to the number of variations in sign of $f(-x)$ or less by even integer.

- Describe the possible real zeros of

$$f(x) = -2x^3 + 5x^2 - x + 8$$

Positive: 3 or 1

Negative: $f(-x) = -2(-x)^3 + 5(-x)^2 - (-x) + 8$

$$f(-x) = 2x^3 + 5x^2 + x + 8$$

0

- Complex Conjugate Theorem
  - If a complex number $a + bi$ is a zero, then $a - bi$ is also a zero.

- Find a polynomial with real coefficients with zeros $\frac{2}{3}, -1, 3+\sqrt{2}i$

$$(3x-2)(x+1)\left(x-\left(3+\sqrt{2}i\right)\right)\left(x-\left(3-\sqrt{2}i\right)\right)$$
$$3x^4-17x^3+25x^2+23x-22$$

2-07 ASYMPTOTES OF RATIONAL FUNCTIONS

In this section, you will:

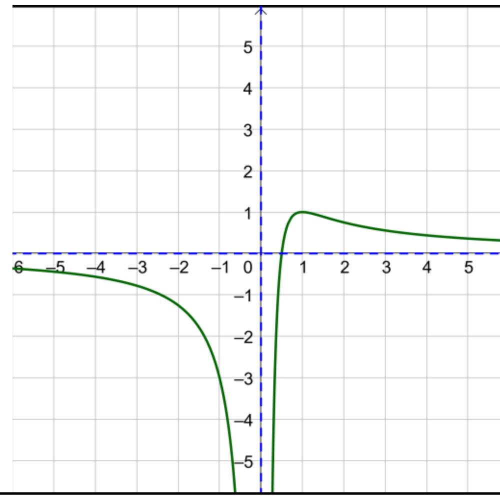
- Find the domains of rational functions.
- Identify vertical asymptotes.
- Identify horizontal asymptotes.

- Rational Function
  - Fraction
- $f(x) = \frac{2x+1}{3x-4}$

- Domain:
  - Denominator ≠ 0

3x-4≠0
X≠4/3
(-∞, 4/3) U (4/3, ∞)

• Asymptotes describe behavior of the graph at the edges



- Vertical Asymptotes
  - Factor and reduce
  - Set denominator = 0 and solve for *x*
- Horizontal Asymptotes
  - Plug in huge number for *x* and simplify
  - OR
  - Find degree of numerator (N) and denominator(D)
  - If N<D, y = 0
  - If N=D, y = leading coeff
  - If N>D, No HA

- Find the asymptotes of $f(x) = \frac{5x^2}{x^2-1}$

VA: $x^2 - 1 = 0 \rightarrow x = -1, 1$
HA: $N = 2, D = 2$ $\rightarrow$ leading coefficients y=5

- For $f(x) = \frac{2x^2 - x}{2x^2 + x - 1}$
  a. Find the domain
  b. Find the removable discontinuity
  c. Find the asymptotes

Domain: $2x^2 + x - 1 \neq 0 \rightarrow (2x-1)(x+1) \neq 0 \rightarrow x \neq \frac{1}{2}, -1$

Discontinuity: Factor $\frac{x(2x-1)}{(2x-1)(x+1)} = \frac{x}{x+1}$ → (2x-1) cancels → $x = \frac{1}{2}$ → get point from reduced function $\left(\frac{1}{2}, \frac{1}{3}\right)$

Asymptotes: VA: x=-1; HA: y=1

- Slant Asymptote
  - If $N = D + 1$, Divide and ignore remainder
- Find the asymptotes of $f(x) = \frac{3x^2+1}{x}$

VA: x=0

HA: N=2, D=1 → N>D → no HA

SA: $(3x^2 + 0x + 1) \div (x + 0) = 3x + \frac{1}{x}$ → y=3x

2-08 GRAPHS OF RATIONAL FUNCTIONS

In this section, you will:

- Find the intercepts of rational functions.
- Graph rational functions.
- Solve applied problems involving rational functions.

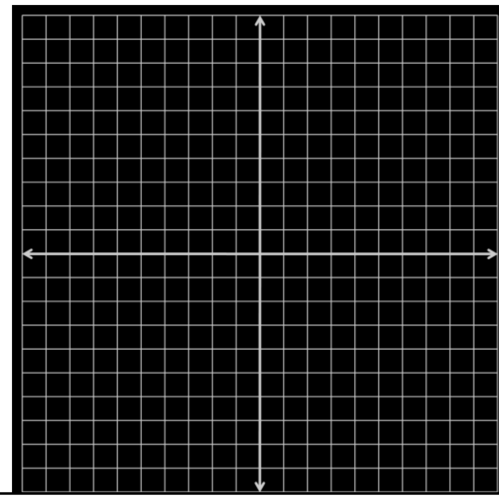
- Intercepts
  - x-int: let y = 0
    - Numerator = 0
  - y-int: let x = 0

- Find the intercepts of $f(x) = \frac{3x^2-1}{x}$

X-int: $\pm\frac{\sqrt{3}}{3}$
Y-int: none

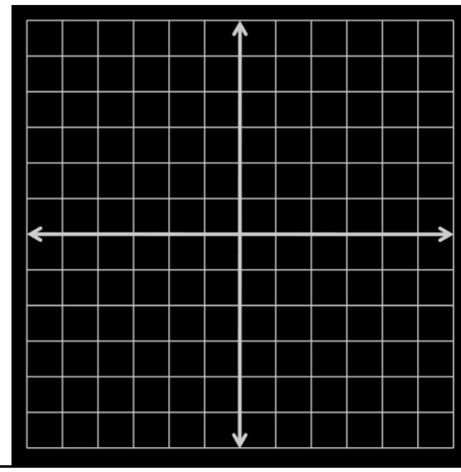
- To graph rational functions
  1. Find domain
  2. Find asymptotes
  3. Graph asymptotes as dotted lines
  4. Create table of values around vertical asymptotes
  5. Plot points
  6. Draw curves starting near an asymptote and ending near another asymptote
     Don't cross VA
  7. Put any required holes.
     Check the domain

- Graph $f(x) = \frac{3x^2+1}{x}$



Domain: x ≠ 0
VA: x = 0
HA: none
SA: y = 3x

- Graph $f(x) = \frac{3x}{x^2+x-2}$

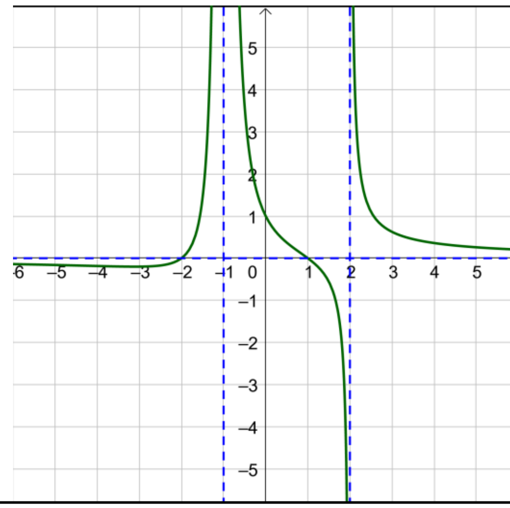


Factor: $\frac{3x}{(x+2)(x-1)}$
Domain: x ≠ −2, 1
VA: x = −2, x = 1
HA: y = 0

- Find the function given a graph

1. Use the x-intercepts and multiplicity to get factors of numerator
   a. If cross x-axis: multiplicity 1 or 3
   b. If touch but not cross: multiplicity 2 or 4
2. Use VA to get factors of denominator
   a. If 1 end goes up and 1 down: multiplicity 1
   b. If both ends go same direction: multiplicity 2
3. Use any other point to get stretch factor, *a*

- Find the function



Numerator → X-ints: -2 and 1 with multiplicity 1 each. Factors of numerator are (x+2)(x-1)

Denominator → VA: x=-1 with multiplicity 2 and x=2 with multiplicity 1. Factors of denominator are $(x+1)^2(x-2)$

Function is $f(x) = a\frac{(x+2)(x-1)}{(x+1)^2(x-2)}$

Use point (0,1) to find stretch factor *a*

$$1 = a\frac{(0+2)(0-1)}{(0+1)^2(0-2)}$$
$$1 = a(1)$$
$$a = 1$$

Function is $f(x) = \frac{(x+2)(x-1)}{(x+1)^2(x-2)}$

2-09 NONLINEAR INEQUALITIES

In this section, you will:

- Find critical numbers of nonlinear inequalities.
- Solve one-variable nonlinear inequalities algebraically.
- Solve one-variable nonlinear inequalities by graphing.

- Steps to solve
1. Make one side 0
2. Find critical #'s
   - Zeros
   - Undefined
3. Plot on #-line
4. Pick test points and test them

- Solve $x^2 - 5x - 6 < 0$

Zeros: 6, -1 (no undefined)
Test points: -2 (F), 0 (T), 7 (F)
Answer: -1<x<6 or (-1, 6)

Also solve by showing graph overlaid on number line

- Solve $3x^3 - 4x^2 - 12x > -16$

Make 0 on right side
Factor by grouping
Critical numbers:
  zeros: 4/3, 2, -2
  no undefined
Test points -3 (F), 0(T), 3/2(F), 3 (T)
Answer: $\left(-2, \frac{4}{3}\right)$ U $(2, \infty)$

Also solve by showing graph overlaid on number line

- Solve $\frac{3x-5}{x-3} \leq 1$

Make right side 0 (use common denominator (x-3) to combine fractions

$$\frac{3x-5}{x-3} - \frac{x-3}{x-3} \leq 0$$

$$\frac{2x-2}{x-3} \leq 0$$

Critical numbers

zeros: 1 (can be equal to)

undefined: 3 (cannot be equal to)

Test points: 0 (F), 2(T), 4(F)

Answer: [1, 3)

Also solve by showing graph overlaid on number line